

ILL-POSEDNESS OF THE NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE IN 3D

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ABSTRACT. We prove that the Cauchy problem for the three dimensional Navier-Stokes equations is ill posed in $\dot{B}_{\infty}^{-1,\infty}$ in the sense that a “norm inflation” happens in finite time. More precisely, we show that initial data in the Schwartz class \mathcal{S} that are arbitrarily small in $\dot{B}_{\infty}^{-1,\infty}$ can produce solutions arbitrarily large in $\dot{B}_{\infty}^{-1,\infty}$ after an arbitrarily short time. Such a result implies that the solution map itself is discontinuous in $\dot{B}_{\infty}^{-1,\infty}$ at the origin.

1. INTRODUCTION

In this paper we address a long standing open problem concerning well-posedness of the three dimensional Navier-Stokes equations in the largest critical space $\dot{B}_{\infty}^{-1,\infty}$ and prove that the Cauchy problem for the three dimensional Navier-Stokes equations is ill posed in $\dot{B}_{\infty}^{-1,\infty}$.

The Navier-Stokes equations for the incompressible fluid in \mathbb{R}^3 are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \quad (1.1)$$

$$\nabla \cdot u = 0, \quad (1.2)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad (1.3)$$

for the unknown velocity vector $u = u(x, t) \in \mathbb{R}^3$ and the pressure $p = p(x, t) \in \mathbb{R}$, where $x \in \mathbb{R}^3$ and $t \in [0, \infty)$.

We adapt the standard notion of well-posedness. More precisely, a Cauchy problem is said to be *locally well-posed* in Z if for every initial data $u_0(x) \in Z$ there exists a time $T = T(\|u_0\|_Z) > 0$ such that a solution to the initial value problem exists in the time interval $[0, T]$, is unique in a certain Banach space of functions $Y \subset C([0, T]; Z)$ and the solution map from the initial data u_0 to the solution u is continuous from Z to $C([0, T]; Z)$. If T can be taken

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arbitrarily large we say that the Cauchy problem is *globally well-posed*. Also we say that the Cauchy problem is *ill-posed* if it is not well-posed. Having such a definition of ill-posedness it is clear that the problem may be ill-posed due to different reasons ranging from a failure of a solution map to be continuous to a more serious type of ill-posedness such as a blow-up in finite time. Here we shall establish an ill-posedness of the Navier-Stokes initial value problem (1.1) - (1.3) via proving a finite time blow-up for solutions to the Navier-Stokes equations in the largest critical space, the Besov space $\dot{B}_{\infty}^{-1,\infty}$.

In order to understand the role of the space $\dot{B}_{\infty}^{-1,\infty}$ in the analysis of the Navier-Stokes equations we recall the scaling property of the equations first. It is easy to see that if the pair $(u(x, t), p(x, t))$ solves the Navier-Stokes equations (1.1) then $(u_{\lambda}(x, t), p_{\lambda}(x, t))$ with

$$\begin{aligned} u_{\lambda}(x, t) &= \lambda u(\lambda x, \lambda^2 t), \\ p_{\lambda}(x, t) &= \lambda^2 p(\lambda x, \lambda^2 t) \end{aligned}$$

is a solution to the system (1.1) with the initial data

$$u_0_{\lambda} = \lambda u_0(\lambda x) .$$

The spaces which are invariant under such a scaling are called critical spaces for the Navier-Stokes. Examples of critical spaces for the Navier-Stokes in 3D are:

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow \dot{B}_{p|p<\infty}^{-1+\frac{3}{p},\infty} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}_{\infty}^{-1,\infty}. \quad (1.4)$$

Kato [9] initiated the study of the Navier-Stokes equations in critical spaces by proving that the problem (1.1)-(1.3) is locally well-posed in L^3 and globally well-posed if the initial data are small in $L^3(\mathbb{R}^3)$. The study of the Navier-Stokes equations in critical spaces was continued by many authors, see, for example, [8, 17, 2, 16]. In particular, in 2001 Koch and Tataru [12] proved the global well-posedness of the Navier-Stokes equations evolving from small initial data in the space BMO^{-1} . The space BMO^{-1} has a special role since it is the largest critical space among the spaces listed in (1.4) where such existence results are available.

The importance of considering the three dimensional Navier-Stokes equations in the Besov space $\dot{B}_{\infty}^{-1,\infty}$ is related to the fact that all critical spaces for the 3D Navier-Stokes equations are embedded in the same function space, $\dot{B}_{\infty}^{-1,\infty}$. A proof of this embedding could be found in, for example, [3]. It has been a long standing problem to determine if the Navier-Stokes initial value problem is well-posed in the space $\dot{B}_{\infty}^{-1,\infty}$. The problem is stated as a conjecture in [3] and [14].

An indication that the Navier-Stokes initial value problem might be ill-posed in the largest critical space is given in [15], where Montgomery-Smith proved

a finite time blow-up for solutions of a simplified model for the Navier-Stokes equations in the space $\dot{B}_{\infty}^{-1,\infty}$. The work [15] suggests that the applications of a fixed point argument that are available up to now are not likely to produce an existence result for the Navier-Stokes equations themselves in the largest critical space, but it does not prove this for the actual Navier-Stokes equations.

In this paper we prove that the actual Navier-Stokes system is ill-posed in $\dot{B}_{\infty}^{-1,\infty}$ in the sense that there is a so called “norm inflation” (for similar results in the context of NLS see, e.g. [5]). Here by a “norm inflation” we mean that initial data in the Schwartz class \mathcal{S} that are arbitrarily small in $\dot{B}_{\infty}^{-1,\infty}$ can produce solutions arbitrarily large in $\dot{B}_{\infty}^{-1,\infty}$ after an arbitrarily short time. Such a result implies that the solution map itself is discontinuous in $\dot{B}_{\infty}^{-1,\infty}$ at the origin. More precisely, our “norm inflation” result can be formulated in the following way:

Theorem 1.1. *For any $\delta > 0$ there exists a solution (u, p) to the Navier-Stokes equations (1.1) - (1.3) and $0 < t < \delta$ such that $u(0) \in \mathcal{S}$*

$$\|u(0)\|_{\dot{B}_{\infty}^{-1,\infty}} \leq \delta,$$

with

$$\|u(t)\|_{\dot{B}_{\infty}^{-1,\infty}} > \frac{1}{\delta}.$$

We remark that similar programs of establishing ill-posedness have been successfully carried out in the context of the nonlinear dispersive equations, see for example work of Bourgain [1], Kenig, Ponce, Vega [11], Christ-Colliander-Tao [5], [6].

The main idea of our approach is to choose initial data u_0 in $\dot{B}_{\infty}^{-1,\infty} \cap \mathcal{S}$ so that when they evolve in time a certain part of the solution will become arbitrarily large in finite time. More precisely, we write a solution to the Navier-Stokes equations (1.1) - (1.3) as

$$u = e^{t\Delta}u_0 - u_1 + y,$$

where u_1 is the first approximation of the solution to the corresponding linear equation and is given by

$$u_1(x, t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(e^{\tau\Delta}u_0 \cdot \nabla) e^{\tau\Delta}u_0 \, d\tau,$$

where \mathbb{P} denotes the projection on divergence free vector fields. We decompose u_1 as $u_1 = u_{1,0} + u_{1,1}$, so that the piece $u_{1,0}$ gets arbitrarily large in finite time. On the other hand, we obtain a PDE that y solves, thanks to which we control $e^{t\Delta}u_0 - u_{1,1} + y$ in the space X_T that was introduced in [12] by Koch and Tataru (see Section 2 for a precise definition of X_T).

We note that recently Chemin and Gallagher [4] established global existence of solutions for the Navier-Stokes equations evolving from arbitrary large initial data in $\dot{B}_{\infty}^{-1,\infty}$ under the assumption of a certain nonlinear smallness on the initial data. Since the initial data that we exhibit do not appear to satisfy this nonlinear smallness condition, our work could be understood as a complement of [4].

After we completed the present paper we learned about the recent work of Germain [7] where he proves an instability result for the Navier-Stokes equations in $\dot{B}_{\infty}^{-1,q}$, for $q > 2$ by showing that the map from the initial data to the solution is not in the class \mathcal{C}^2 . We remark that [7] does not treat a norm inflation phenomenon.

Organization of the paper. In section 2 we introduce the notation that shall be used throughout the paper. Also in Section 2 we recall the result of Koch and Tataru [12]. In section 3 we present a proof of Theorem 1.1.

2. PRELIMINARIES

2.1. Notation. We shall denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some constant C . Throughout the paper, i^{th} coordinate ($i = 1, 2, 3$) of a vector $x \in \mathbb{R}^3$ will be denoted by x^i .

We recall that the Besov space $\dot{B}_{\infty}^{-1,\infty}$ is equipped with the norm

$$\|f(\cdot)\|_{\dot{B}_{\infty}^{-1,\infty}} = \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} f(\cdot)\|_{L^{\infty}}.$$

2.2. The result of Koch and Tataru. Here we recall the result of Koch and Tataru [12] that establishes the global well-posedness of the Navier-Stokes equations evolving from small initial data in the space BMO^{-1} .

First, let us recall the definition of the space BMO^{-1} as given in [12]:

$$\|f(\cdot)\|_{BMO^{-1}} = \sup_{x_0, R} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |e^{t\Delta} f(y)|^2 dy dt \right)^{\frac{1}{2}}. \quad (2.1)$$

In [12] Koch and Tataru proved the following existence theorem:

Theorem 2.1. *The Navier-Stokes equations (1.1) - (1.3) have a unique global solution in X*

$$\begin{aligned} \|u(\cdot, \cdot)\|_X &= \sup_t t^{\frac{1}{2}} \|u(\cdot, t)\|_{L^\infty} \\ &+ \sup_{x_0, R} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |u(y, t)|^2 dy dt \right)^{\frac{1}{2}}, \end{aligned}$$

for all initial data u_0 with $\nabla \cdot u_0 = 0$ which are small in BMO^{-1} .

Let $T \in (0, \infty]$. We denote by X_T the space equipped with the norm

$$\begin{aligned} \|u(\cdot, \cdot)\|_{X_T} &= \sup_{0 < t < T} t^{\frac{1}{2}} \|u(\cdot, t)\|_{L^\infty} \\ &+ \sup_{x_0} \sup_{0 < R < T} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |u(y, t)|^2 dy dt \right)^{\frac{1}{2}}. \end{aligned}$$

Now let \mathbb{P} denote the projection on divergence free vector fields. As shown in [12], see also [13], the bilinear operator

$$\mathcal{B}(u, v) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(u \cdot \nabla) v d\tau, \quad (2.2)$$

maps $X_T \times X_T$ into X_T . More precisely,

$$\|\mathcal{B}(u, v)\|_{X_T} \lesssim \|u\|_{X_T} \|v\|_{X_T}. \quad (2.3)$$

3. PROOF OF THEOREM 1.1

We rewrite the Navier-Stokes equations (1.1) in the following way:

$$u = e^{t\Delta} u_0 - u_1 + y, \quad (3.1)$$

where

$$u_1(x, t) = \mathcal{B}(e^{t\Delta} u_0(x), e^{t\Delta} u_0(x)), \quad (3.2)$$

and y satisfies the following equation:

$$\partial_t y - \Delta y + G_1 + G_2 + G_3 = 0,$$

where

$$\begin{aligned} G_1 &= \mathbb{P}[(e^{t\Delta} u_0 \cdot \nabla) y + (u_1 \cdot \nabla) y + (y \cdot \nabla) e^{t\Delta} u_0 + (y \cdot \nabla) u_1] \\ G_2 &= \mathbb{P}[(y \cdot \nabla) y] \\ G_3 &= \mathbb{P}[(e^{t\Delta} u_0 \cdot \nabla) u_1 + (u_1 \cdot \nabla) e^{t\Delta} u_0 + (u_1 \cdot \nabla) u_1]. \end{aligned} \quad (3.3)$$

We shall choose initial data u_0 in such a way that when they evolve in time, the part of the solution u_1 will become arbitrarily large in $\dot{B}_{\infty}^{-1,\infty}$ at certain time T , while we will be able to control the behavior of y in the space X_T .

3.1. Choice of initial data. Fix small numbers $T > 0$, $\delta > 0$ and a large number $Q > 0$ (eventually $T \rightarrow 0$, $\delta \rightarrow 0$ and $Q \rightarrow \infty$). Let $\eta \in \mathbb{S}^2$. Let $r = r(Q)$ be a large integer (to be specified). We choose the initial data as follows:

$$u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^r |k_s| [v_s \cos(k_s \cdot x) + v'_s \cos(k'_s \cdot x)], \quad (3.4)$$

where

- (1) The vectors $k_s \in \mathbb{R}^3$ are parallel to a given vector $k_0 \in \mathbb{R}^3$ and $k'_s \in \mathbb{R}^3$ is defined by

$$k_s - k'_s = \eta. \quad (3.5)$$

Furthermore, we take $|k_0|$ large (depending on Q) and $|k_s|$ ($1 \leq s \leq r$) very lacunary. For example,

$$|k_s| = 2^s |k_0| |k_{s-1}|, \quad s = 2, 3, \dots, r.$$

- (2) $v_s, v'_s \in \mathbb{S}^2$ such that
(a)

$$k_s \cdot v_s = 0 = k'_s \cdot v'_s. \quad (3.6)$$

Note that (3.6) implies that $\operatorname{div} u_0 = 0$.

- (b) By (3.5) we may ensure that

$$v_s \approx v'_s \approx v \in \mathbb{S}^2.$$

We require that

$$\eta \cdot v_s = \eta \cdot v'_s = \eta \cdot v = \frac{1}{2}. \quad (3.7)$$

It is obvious from (3.4) that

$$\|u_0\|_{\dot{B}_{\infty}^{-1,\infty}} \sim \frac{Q}{\sqrt{r}} < \delta$$

for appropriate r .

3.2. Analysis of u_1 . Now we analyze u_1 with a goal to split it into two pieces $u_{1,0}$ and $u_{1,1}$ such that the upper bound on $u_{1,0}$ in the Besov space $\dot{B}_{\infty}^{-1,\infty}$ is roughly Q^2 on a certain time interval.

For the initial data u_0 given by (3.4), $e^{\tau\Delta}u_0$ can be written as follows

$$e^{\tau\Delta}u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^r |k_s| \left(v_s \cos(k_s \cdot x) e^{-|k_s|^2\tau} + v'_s \cos(k'_s \cdot x) e^{-|k'_s|^2\tau} \right). \quad (3.8)$$

Hence we can calculate $(e^{\tau\Delta}u_0 \cdot \nabla) e^{\tau\Delta}u_0$ via its coordinates as follows:

$$\begin{aligned} ((e^{\tau\Delta}u_0 \cdot \nabla) e^{\tau\Delta}u_0)^i &= \sum_j \partial_j [(e^{\tau\Delta}u_0)^i (e^{\tau\Delta}u_0)^j] \\ &\sim N_1^i + N_2^i + N_3^i, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} N_1^i &= \frac{Q^2}{r} \sum_{s=1}^r |k_s|^2 e^{-2|k_s|^2\tau} \sin(\eta \cdot x) [(\eta \cdot v'_s) v_s^i + (\eta \cdot v_s) (v'_s)^i] \\ N_2^i &= \frac{Q^2}{r} \sum_{s=1}^r |k_s|^2 e^{-(|k_s|^2 + |k'_s|^2)\tau} \sin((k_s + k'_s) \cdot x) \times \\ &\quad \times [((k_s + k'_s) \cdot v'_s) v_s^i + ((k_s + k'_s) \cdot v_s) (v'_s)^i] \\ N_3^i &= \frac{Q^2}{r} \sum_{s \neq s'} |k_s| |k_{s'}| e^{-(|k_s|^2 + |k_{s'}|^2)\tau} \sin((k_s \pm k_{s'}) \cdot x) \times \\ &\quad \times [((k_s \pm k_{s'}) \cdot v_{s'}) v_s^i + ((k_s \pm k_{s'}) \cdot v_s) v_{s'}^i] + \text{similar terms} . \end{aligned}$$

We consider contributions to u_1 coming from each of three terms N_1 , N_2 , N_3 . Contributions coming from N_1 can be estimated by integrating in time and using (3.7) as follows

$$\begin{aligned} &\int_0^t e^{(t-\tau)\Delta} N_1 \, dt \\ &\sim \frac{Q^2}{r} \sum_{s=1}^r |k_s|^2 \left[\int_0^t e^{-(t-\tau)|\eta|^2 - 2|k_s|^2\tau} \, d\tau \right] \sin(\eta \cdot x) [(\eta \cdot v'_s) v_s + (\eta \cdot v_s) v'_s] \\ &\sim Q^2 \sin(\eta \cdot x) v, \end{aligned}$$

for

$$\frac{1}{|k_1|^2} \ll T \ll 1. \quad (3.10)$$

Therefore, recalling (3.7)

$$\left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(N_1) \, dt \right\|_{B_{\infty}^{-1,\infty}} \sim Q^2. \quad (3.11)$$

Also

$$\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(N_1) dt \|_{X_T} \lesssim \sqrt{T} Q^2. \quad (3.12)$$

Now consider contributions to u_1 coming from N_3 .

$$\begin{aligned} & \left| \int_0^t e^{(t-\tau)\Delta} N_3 dt \right| \\ & \lesssim \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_{s'}| \left| \int_0^t e^{-(t-\tau)|k_s \pm k_{s'}|^2 - (|k_s|^2 + |k_{s'}|^2)\tau} d\tau \right| \times \\ & \quad \times |\sin((k_s \pm k_{s'}) \cdot x)| O(|k_s|) \\ & \sim \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_{s'}| \left| \frac{e^{-(|k_s|^2 + |k_{s'}|^2)t} - e^{-|k_s \pm k_{s'}|^2 t}}{|k_s \pm k_{s'}|^2 - (|k_s|^2 + |k_{s'}|^2)} \right| O(|k_s|) \\ & \lesssim \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_{s'}| e^{-\frac{1}{2}|k_{s'}|^2 t} t O(|k_s|) \end{aligned} \quad (3.13)$$

$$\lesssim \frac{Q^2}{r} \sum_{s=1}^r |k_{s-1}| e^{-\frac{1}{l} k_s^2 t}, \quad (3.14)$$

where to obtain (3.13) we use the boundedness of the function $g(t) = \frac{1-e^{-\lambda t}}{\lambda t}$, with $\lambda > 0$, while to obtain (3.14) we use the boundedness of the function $h(t) = \mu t e^{-\mu t}$, with $\mu > 0$ and we replace $e^{-k_{s'}^2}$ by $e^{-\frac{1}{l} k_s^2}$ for some l . We also use the lacunarity of the sequence $|k_s|$.

Thus (3.14) implies that

$$\begin{aligned} & \| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(N_3) dt \|_{X_T} \\ & \lesssim \frac{Q^2}{r} \sum_{s=1}^r \frac{|k_{s-1}|}{|k_s|} + \frac{Q^2}{r} \sup_{t < T} \left\{ \int_0^t \left[\sum_{s=1}^r |k_{s-1}| e^{-\frac{1}{l} |k_s|^2 \tau} \right]^2 d\tau \right\}^{\frac{1}{2}} \\ & \lesssim \frac{Q^2}{r} \sum_{s=1}^r \frac{|k_{s-1}|}{|k_s|} \\ & < \frac{Q^2}{r}, \end{aligned} \quad (3.15)$$

again by lacunarity of $|k_s|$.

Next we estimate the contribution coming from N_2 . Clearly, recalling (3.5)

$$\int_0^t e^{(t-\tau)\Delta} N_2 dt \sim \frac{Q^2}{r} \left\{ \sum_{s=1}^r O(|k_s| e^{-|k_s|^2 t}) \sin(k_s + k'_s) \cdot x \right\}.$$

Therefore

$$\begin{aligned}
& \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(N_2) d\tau \right\|_{X_T} \\
& \lesssim \frac{Q^2}{r} \sup_{t>0} \left| \sum_{s=1}^r t^{\frac{1}{2}} |k_s| e^{-|k_s|^2 t} \right| \\
& + \frac{Q^2}{r} \sup_{R>0} \left\{ \int_0^R \left[\sum_{|k_s| > \frac{1}{\sqrt{R}}} |k_s|^2 e^{-|k_s|^2 t} \right] dt + \int_0^R \left(\sum_{|k_s| \leq \frac{1}{\sqrt{R}}} |k_s| \right)^2 dt \right\}^{\frac{1}{2}} \\
& \lesssim \frac{Q^2}{r} + \frac{Q^2}{r} (r+1)^{\frac{1}{2}} \\
& \lesssim \frac{Q^2}{\sqrt{r}}, \tag{3.16}
\end{aligned}$$

using the fact that $\sqrt{t} \sum_s |k_s|^2 e^{-|k_s|^2 t} \lesssim 1$ and making the appropriate splitting to bound the second term in $\|\cdot\|_{X^T}$.

Hence we can decompose u_1 as follows

$$u_1 = u_{1,0} + u_{1,1},$$

where

$$\begin{aligned}
& \|u_{1,0}\|_{B_{\infty}^{-1,\infty}} \sim Q^2 \text{ for } \frac{1}{|k_1|^2} \ll t \ll 1, \\
& \|u_{1,0}\|_{X_T} \lesssim \sqrt{T} Q^2, \\
& \|u_{1,1}\|_{X_T} \lesssim \frac{Q^2}{\sqrt{r}}. \tag{3.17}
\end{aligned}$$

3.3. Analysis of y . Now we analyze the remaining part of the solution, which we denoted by y . The main idea is to control y using the space of Koch and Tataru X_T .

Consider time-intervals

$$0 < T_1 < T_2 < \dots < T_\beta, \quad \beta = Q^3$$

with

$$T_\alpha^{-1} = |k_{r_\alpha}|^2 \tag{3.18}$$

$$r_\alpha = r - \alpha Q^{-3} r, \quad \alpha = 1, 2, \dots \tag{3.19}$$

In particular, $r_\beta = 0$ and $T_\beta^{-1} = |k_0|^2$.

For $t \geq T_\alpha$ the equation for y can be written in the integral form as

$$y(t) = e^{(t-T_\alpha)\Delta} y(T_\alpha) - \int_{T_\alpha}^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) d\tau, \quad (3.20)$$

where G_i , $i = 1, 2, 3$ are given by (3.3).

Also

$$y(T_\alpha) = \int_0^{T_\alpha} e^{(T_\alpha-\tau)\Delta} [G_1 + G_2 + G_3](\tau) d\tau.$$

Therefore

$$\begin{aligned} e^{(t-T_\alpha)\Delta} y(T_\alpha) &= \int_0^{T_\alpha} e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) d\tau \\ &= \int_0^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) \chi_{[0, T_\alpha]} d\tau, \end{aligned} \quad (3.21)$$

where $\chi_{[0, T_\alpha]}$ is a characteristic function of the interval $[0, T_\alpha]$.

Now we substitute (3.21) in (3.20) to obtain

$$\|y\|_{X_{T_{\alpha+1}}} \leq I + II, \quad (3.22)$$

where

$$I = \left\| \int_0^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) \chi_{[0, T_\alpha]}(\tau) d\tau \right\|_{X_{T_{\alpha+1}}} \quad (3.23)$$

and

$$II = \left\| \int_0^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) \chi_{[T_\alpha, T_{\alpha+1}]}(\tau) d\tau \right\|_{X_{T_{\alpha+1}}}. \quad (3.24)$$

Next we use the bilinear estimate (2.3) on the terms in G_1 , G_2 and G_3 to obtain an upper bound on I and II respectively. Before we obtain an upper bound on I , we estimate $\|e^{t\Delta} u_0\|_{X_{T_\alpha}}$. From (3.8) we have

$$e^{t\Delta} u_0 \approx \frac{Q}{\sqrt{r}} \sum_{s \leq r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t}.$$

We estimate $\|e^{t\Delta}u_0\|_{X_{T_\alpha}}$ as follows

$$\|e^{t\Delta}u_0\|_{X_{T_\alpha}} \leq \frac{Q}{\sqrt{r}} \sup_{t < T_\alpha} \sqrt{t} \sum_{s \leq r} |k_s| e^{-k_s^2 t} \quad (3.25)$$

$$\begin{aligned} & + \frac{Q}{\sqrt{r}} \sup_{x_0, 0 < t < T_\alpha} \left(t^{-3/2} \int_0^t \int_{|x-x_0| < \sqrt{t}} \left| \sum_{s \leq r} |k_s| v_s \cos(k_s \cdot x) e^{-k_s^2 \tau} \right|^2 dx d\tau \right)^{1/2} \\ & \lesssim \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r+1)^{\frac{1}{2}} \\ & \lesssim Q, \end{aligned} \quad (3.26)$$

similarly to (3.16). Hence

$$\|e^{t\Delta}u_0\|_{X_{T_\alpha}} \lesssim Q. \quad (3.27)$$

Now we are ready to estimate I using (3.3) and the bilinear estimate (2.3):

$$\begin{aligned} I & \lesssim (\|e^{t\Delta}u_0\|_{X_{T_\alpha}} + \|u_1\|_{X_{T_\alpha}} + \|y\|_{X_{T_\alpha}}) \|y\|_{X_{T_\alpha}} \\ & \quad + (\|e^{t\Delta}u_0\|_{X_{T_\alpha}} + \|u_1\|_{X_{T_\alpha}}) \|u_1\|_{X_{T_\alpha}} \\ & \leq \left(Q + Q^2 T_\alpha^{1/2} + \frac{Q^2}{\sqrt{r}} + \|y\|_{X_{T_\alpha}} \right) \|y\|_{X_{T_\alpha}} \\ & \quad + \left(Q + Q^2 T_\alpha^{1/2} + \frac{Q^2}{\sqrt{r}} \right) \left(Q^2 T_\alpha^{1/2} + \frac{Q^2}{\sqrt{r}} \right), \end{aligned} \quad (3.28)$$

where to obtain (3.28) we used (3.27) and (3.17).

In order to obtain an upper bound on II , first, we estimate $\|(e^{t\Delta}u_0) \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}}$. More precisely, from (3.8) we have

$$(e^{t\Delta}u_0) \chi_{[T_\alpha, T_{\alpha+1}]}(t) \approx L_1 + L_2, \quad (3.29)$$

where

$$L_1 = \frac{Q}{\sqrt{r}} \sum_{s < r_{\alpha+1}} |k_s| v_s \cos(k_s \cdot x) \chi_{[T_\alpha, T_{\alpha+1}]}(t)$$

and

$$L_2 = \frac{Q}{\sqrt{r}} \sum_{s=r_{\alpha+1}}^{r_\alpha} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha, T_{\alpha+1}]}(t).$$

We estimate L_1 keeping in mind that, thanks to (3.18), $T_{\alpha+1} = |k_{r_{\alpha+1}}|^{-2}$:

$$\begin{aligned}
\|L_1\|_{X_{T_{\alpha+1}}} &\leq \frac{Q}{\sqrt{r}} T_{\alpha+1}^{1/2} |k_{r_{\alpha+1}-1}| \\
&\quad + \frac{Q}{\sqrt{r}} \sup_{x_0, t} \left(t^{-3/2} \int_0^t \int_{|x-x_0| < \sqrt{t}} \left| \sum_{s < r_{\alpha+1}} |k_s| \chi_{[T_\alpha, T_{\alpha+1}]}(\tau) \right|^2 dx d\tau \right)^{1/2} \\
&\leq \frac{Q}{\sqrt{r}} \frac{|k_{r_{\alpha+1}-1}|}{|k_{r_{\alpha+1}}|} + \frac{Q}{\sqrt{r}} (T_{\alpha+1} |k_{r_{\alpha+1}-1}|^2)^{1/2} \\
&< \frac{Q}{\sqrt{r}}.
\end{aligned} \tag{3.30}$$

We estimate L_2 as follows

$$\begin{aligned}
\|L_2\|_{X_{T_{\alpha+1}}} &\leq \frac{Q}{\sqrt{r}} \sup_t \sqrt{t} \sum_{s=r_{\alpha+1}}^{r_\alpha} |k_s| e^{-k_s^2 t} \\
&\quad + \frac{Q}{\sqrt{r}} \sup_{x_0, t} \left(t^{-3/2} \int_0^t \int_{|x-x_0| < \sqrt{t}} \left| \sum_{s=r_{\alpha+1}}^{r_\alpha} |k_s| v_s \cos(k_s \cdot x) e^{-k_s^2 \tau} \chi_{[T_\alpha, T_{\alpha+1}]}(\tau) \right|^2 dx d\tau \right)^{1/2} \\
&\leq \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r_\alpha - r_{\alpha+1})^{1/2} \\
&= \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (Q^{-3} r)^{1/2} \\
&\leq Q^{-1/2},
\end{aligned} \tag{3.31}$$

where to obtain (3.31) we used (3.19). Hence we combine (3.29), (3.30) and (3.32) to conclude

$$\|(e^{t\Delta} u_0) \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2}. \tag{3.33}$$

Also we recall that (3.17) implies

$$\|u_1 \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{r}}. \tag{3.34}$$

Now we are ready to find an upper bound on II by employing the bilinear estimate (2.3):

$$\begin{aligned}
II &\lesssim \left(\|(e^{t\Delta} u_0) \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} + \|u_1 \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} + \|y\|_{X_{T_{\alpha+1}}} \right) \|y\|_{X_{T_{\alpha+1}}} \\
&\quad + \left(\|(e^{t\Delta} u_0) \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} + \|u_1 \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \right) \|u_1 \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \\
&\leq \left(Q^{-1/2} + Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{r}} + \|y\|_{X_{T_{\alpha+1}}} \right) \|y\|_{X_{T_{\alpha+1}}} \\
&\quad + \left(Q^{-1/2} + Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{r}} \right) \left(Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{r}} \right), \tag{3.35}
\end{aligned}$$

where to obtain (3.35) we used (3.33) and (3.34).

Having in mind that $T_\alpha < T_{\alpha+1} < T$ and that T will be chosen to satisfy (3.45), we combine (3.22), (3.28) and (3.35) to obtain

$$\|y\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2} \|y\|_{X_{T_{\alpha+1}}} + \|y\|_{X_{T_{\alpha+1}}}^2 + Q^3 \left(\frac{1}{\sqrt{r}} + T_{\alpha+1}^{1/2} \right) + Q \|y\|_{X_{T_\alpha}}.$$

Thus

$$\|y\|_{X_{T_{\alpha+1}}} \lesssim Q^3 \left(\frac{1}{\sqrt{r}} + T_\beta^{1/2} \right) + Q \|y\|_{X_{T_\alpha}}. \tag{3.36}$$

Iterating (3.36) gives

$$\|y\|_{X_{T_\beta}} \lesssim Q^{\beta+3} \left(\frac{1}{r} + T_\beta \right)^{1/2}. \tag{3.37}$$

Now we take $T > T_\beta$ and write (3.20) and (3.21) with $\alpha = \beta$. Thus

$$\|y\|_{X_T} \leq I_\beta + II_\beta \tag{3.38}$$

where

$$I_\beta = \left\| \int_0^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) \chi_{[0, T_\beta]}(\tau) d\tau \right\|_{X_T} \tag{3.39}$$

and

$$II_\beta = \left\| \int_0^t e^{(t-\tau)\Delta} [G_1 + G_2 + G_3](\tau) \chi_{[T_\beta, T]}(\tau) d\tau \right\|_{X_T}. \tag{3.40}$$

We obtain an upper bound on I_β by using (3.3) and the bilinear estimate (2.3):

$$\begin{aligned} I_\beta &\lesssim \left(\| (e^{t\Delta} u_0) \|_{X_{T_\beta}} + \| u_1 \|_{X_{T_\beta}} + \| y \|_{X_{T_\beta}} \right) \| y \|_{X_{T_\beta}} \\ &\quad + \left(\| (e^{t\Delta} u_0) \|_{X_{T_\beta}} + \| u_1 \|_{X_{T_\beta}} \right) \| u_1 \|_{X_{T_\beta}} \\ &\leq \left(Q + Q^2 T_\beta^{1/2} + \frac{Q^2}{\sqrt{r}} + Q^{Q^3} \left(\frac{1}{r} + T_\beta \right)^{1/2} \right) Q^{Q^3} \left(\frac{1}{r} + T_\beta \right)^{1/2} \end{aligned} \quad (3.41)$$

$$+ \left(Q + Q^2 T_\beta^{1/2} + \frac{Q^2}{\sqrt{r}} \right) \left(Q^2 T_\beta^{1/2} + \frac{Q^2}{\sqrt{r}} \right). \quad (3.42)$$

We rely here on (3.27), (3.17) and (3.37).

Recalling that $T_\beta = |k_0|^{-2}$ and choosing r and $|k_0|$ large enough, it follows from (3.41) and (3.42) that

$$I_\beta \lesssim r^{-1/3} + |k_0|^{-1/2}. \quad (3.43)$$

Also

$$\begin{aligned} II_\beta &\lesssim \left(\| (e^{t\Delta} u_0) \chi_{[T_\beta, T]}(\tau) \|_{X_T} + \| u_1 \|_{X_T} + \| y \|_{X_T} \right) \| y \|_{X_T} \\ &\quad + \left(\| (e^{t\Delta} u_0) \chi_{[T_\beta, T]}(\tau) \|_{X_T} + \| u_1 \|_{X_T} \right) \| u_1 \|_{X_T} \\ &\lesssim \left(|k_1| e^{-\frac{|k_1|^2}{|k_0|^2}} + Q^2 T^{1/2} + \frac{Q^2}{\sqrt{r}} + \| y \|_{X_T} \right) \| y \|_{X_T} + \left(|k_1| e^{-\frac{|k_1|^2}{|k_0|^2}} + Q^2 T^{1/2} + \frac{Q^2}{\sqrt{r}} \right)^2 \end{aligned} \quad (3.44)$$

where to obtain (3.44) we used (3.8) and (3.17).

Let us also assume that

$$T < Q^{-8}. \quad (3.45)$$

Since $|k_1| > |k_0|^2$, (3.44) implies

$$II_\beta < (o(1) + \| y \|_{X_T}) \| y \|_{X_T} + 2Q^4 T. \quad (3.46)$$

Therefore, from (3.43) and (3.46)

$$\| y \|_{X_T} < 3Q^4 T \quad (3.47)$$

implying

$$\| y \|_{L^\infty} \leq T^{-\frac{1}{2}} \| y \|_{X_T} < 3Q^4 T^{\frac{1}{2}}. \quad (3.48)$$

Now we combine (3.1), (3.17) and (3.48) to conclude that

$$\| u(T) - e^{T\Delta} u_0 \|_{\dot{B}_{\infty}^{-1, \infty}} \geq Q^2 - \| u_{1,1} \|_{L^\infty} - \| y \|_{L^\infty} > Q^2 \left(1 - \frac{1}{\sqrt{rT}} - 3Q^2 T^{\frac{1}{2}} \right)$$

and

$$\|u(T)\|_{\dot{B}_{\infty}^{-1,\infty}} > \frac{1}{2}Q^2. \quad (3.49)$$

Consequently we proved that for all $\delta > 0$

$$\sup_{\|u(0)\|_{\dot{B}_{\infty}^{-1,\infty}} \leq \delta} \sup_{0 < t < \delta} \|u(t)\|_{\dot{B}_{\infty}^{-1,\infty}} = \infty. \quad (3.50)$$

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